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SOME PREDICTION PROBLEMS FOR STRICTLY STATIONARY PROCESSES

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1. Introduction

A strictly stationary process $\{x_t\}$ $(-\infty < t < \infty)$ is one whose distributions remain the same as time passes; that is, the multivariate distribution of the random variables $x_{t_1+h}, x_{t_2+h}, \cdots, x_{t_n+h}$ is independent of h. Here $t_1, t_2, \cdots, t_n$ is any finite set of parameter values. Throughout this paper we shall assume that the expectation $E|x_t|$ is finite, $Ex_t = 0$ and $\lim_{h\to 0} E|x_{t+h} - x_t| = 0$. Strictly stationary processes satisfying these additional conditions will be called shortly *stationary*. Moreover, random variables which are equal with probability 1 will be treated here as identical.

Let $[x_t]$ denote the linear space spanned by all random variables x_t $(-\infty < t < \infty)$ and closed with respect to the mean convergence. Of course, $[x_t]$ becomes a Banach space under the norm $\|x\| = E|x|$. Moreover,

(1.1) $$\|x\| \leq \|x + y\| \qquad \text{if } x \text{ and } y \text{ are independent}$$

(see [8], p. 263). It is well known that to each stationary process $\{x_t\}$ there corresponds a unique one-parameter strongly continuous group $\{T_t\}$ of linear operators in $[x_t]$ preserving the probability distribution and such that $x_t = T_t x_0$ (see [1], chapter XI, section 1). Conversely, each such group $\{T_t\}$ in conjunction with a random variable y with $Ey = 0$ defines a stationary process $x_t = T_t y$.

Let $[x_t: t \leq a]$ be the subspace of $[x_t]$ spanned by all random variables x_t with $t \leq a$. We say that a stationary process $\{x_t\}$ *admits a prediction*, if there exists a linear operator A_0 from $[x_t]$ onto $[x_t: t \leq 0]$ such that

(i) $A_0 x = x$ whenever $x \in [x_t: t \leq 0]$,

(ii) if for every $y \in [x_t: t \leq 0]$ the random variables x and y are independent, then $A_0 x = 0$,

(iii) for every $x \in [x_t]$ and $y \in [x_t: t \leq 0]$ the random variables $x - A_0 x$ and y are independent.

The random variable $A_0 x$ can be regarded as a linear prediction of x based on the full past of the process $\{x_t\}$ up to time $t = 0$. An optimality criterion is given by (iii). In what follows the operator A_0 will be called *a predictor* based on the past of the process up to time $t = 0$. The conditions (i), (ii), and (iii) determine the predictor A_0 uniquely.

It should be noted that Gaussian stationary processes with zero mean always admit a prediction. This follows from the fact that in this case the concepts of independence and orthogonality are equivalent, and moreover, the square-mean convergence and the mean convergence are equivalent. Therefore, the predictor A_0 is simply the best linear least squares predictor, that is the orthogonal projector from $[x_t]$ onto $[x_t: t \leq 0]$ (see [1], chapter XII, section 5). Since our stationary processes need not have a finite variance, the problem of prediction discussed in this paper is not contained in the Wiener-Kolmogorov theory of the best linear least squares prediction for wide sense stationary processes. Moreover, the Hilbert space method will be replaced here by a Banach space method.

Let $\{x_t\}$ be a stationary process admitting a prediction. The predictor A_0 and the shift T_a induced by $\{x_t\}$ determine the predictor A_a based on the full past of the process up to time $t = a$. Namely, setting

$$A_a = T_a A_0 T_{-a} \tag{1.2}$$

and taking into account that T_t preserves the probability distribution, and consequently, the independence, we obtain a linear operator from $[x_t]$ onto $[x_t: t \leq a]$ satisfying the following conditions:

(1.3) $A_a x = x$ whenever $x \in [x_t: t \leq a]$;

(1.4) if for every $y \in [x_t: t \leq a]$ the random variables x and y are independent, then $A_a x = 0$;

(1.5) for every $y \in [x_t: t \leq a]$ and $x \in [x_t]$ the random variables $x - A_a x$ and y are independent.

A stationary process $\{x_t\}$ admitting a prediction is called *deterministic*, if $A_0 x = x$ for every $x \in [x_t]$. Further, a stationary process $\{x_t\}$ admitting a prediction is called *completely nondeterministic*, if $\lim_{t \to -\infty} A_t x = 0$ for every $x \in [x_t]$.

The aim of this paper is to prove that any stationary process admitting a prediction can be decomposed into a deterministic component and a completely nondeterministic one. Moreover, we shall give a representation of completely nondeterministic processes by integrals with respect to a stochastic measure. These theorems are an analogue of the well-known Wold's decomposition and representation theorems in the linear least squares prediction theory (see [1], chapter XII, and [4]). Related problems for stationary sequences were considered in [17].

It should be noted that, for a given $x \in [x_t]$, the prediction $A_a x$ furnishes the best approximation of x in the norm $\| \quad \|$ by elements from the subspace $[x_t: t \leq a]$. This fact is a simple consequence of (1.1) and (1.5).

We begin by proving some lemmas from which we deduce the decomposition theorem.

LEMMA 1.1. *For $a \leq b$, the predictors satisfy the equation $A_a = A_a A_b = A_b A_a$.*

PROOF. Since $A_a x \in [x_t: t \leq b]$ for every $x \in [x_t]$, we have, by (1.3), the relation $A_b A_a x = A_a x$. Further, by (1.5), for every $x \in [x_t]$ and $y \in [x_t: t \leq a]$,

the random variables $x - A_b x$ and y are independent. Hence, by (1.4), $A_a x - A_a A_b x = 0$, which completes the proof.

For any bounded semiclosed interval $(a, b]$, we put

$$A((a, b]) = A_b - A_a. \tag{1.6}$$

Moreover, we put $A((-\infty, b]) = A_b$.

LEMMA 1.2. *For every pair J_1, J_2 of intervals, the equation*

$$A(J_1)A(J_2) = A(J_1 \cap J_2) \tag{1.7}$$

holds. Moreover, for any system $I_1, I_2, \cdots, I_n$ of disjoint intervals and $y_1, y_2, \cdots, y_n \in [x_t]$, the random variables $A(I_1)y_1, A(I_2)y_2, \cdots, A(I_n)y_n$ are independent.

PROOF. Formula (1.7) is a simple consequence of lemma 1.1. Suppose that $I_j = (a_j, b_j]$ where

$$-\infty \le a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_n \le b_n. \tag{1.8}$$

For every system $t_1, t_2, \cdots, t_n$ of real numbers we put $z_k = \sum_{j=k+1}^{n} t_j A(I_j)y_j$, $(k = 1, 2, \cdots, n-1)$. From (1.8) and lemma 1.1, we get the formula $A_{b_k} z_k = 0$. Since $A(I_k)y_k \in [x_t: t \le b_k]$, we infer, by (1.5), that the random variables z_k and $A(I_k)y_k$ are independent. Consequently,

$$E \exp\left(i \sum_{j=k}^{n} t_j A(I_j)y_j\right) = E \exp\,(it_k A(I_k)y_k) E \exp\left(i \sum_{j=k+1}^{n} t_j A(I_j)y_j\right). \tag{1.9}$$

Hence, by induction, we get the equation

$$E \exp\left(i \sum_{j=1}^{n} t_j A(I_j)y_j\right) = \prod_{j=1}^{n} E \exp\,(it_j A(I_j)y_j). \tag{1.10}$$

Thus, the multivariate characteristic function of the random variables $A(I_1)y_1$, $A(I_2)y_2, \cdots, A(I_n)y_n$ is equal to the product of their characteristic functions. Hence, we get the independence of these random variables which completes the proof.

LEMMA 1.3. *There exists a linear operator $A_{-\infty}$ on $[x_t]$ commuting with the operations T_t such that for every $x \in [x_t]$, $\lim_{t\to-\infty} A_t x = A_{-\infty}x$.*

PROOF. Given an element $x \in [x_t]$ and a sequence $t_0 > t_1 > t_2 > \cdots$ tending to $-\infty$, we put $I_j = (t_{j-1}, t_j]$ and $z_j = A(I_j)x$, $(j = 1, 2, \cdots)$. By lemma 1.2. the random variables $z_1, z_2, \cdots, z_n$ and $A_{t_n}x$ are independent. Since $A_{t_0}x = \sum_{j=1}^{n} z_j + A_{t_n}x$, and, consequently, by (1.1), $\|\sum_{j=1}^{n} z_j\| \le \|A_{t_0}x\|$, $(n = 1, 2, \cdots)$, the series $\sum_{j=1}^{\infty} z_j$ converges in $[x_t]$ (see [1], p. 338). Hence, it follows that $\lim_{n\to\infty} A_{t_n}x = A_{-\infty}x$ exists. It is clear that $A_{-\infty}$ is a linear operator. Further, from (1.2) we get the formula $T_t A_a = A_{t+a} T_t$ which implies $T_t A_{-\infty} = A_{-\infty} T_t$. The lemma is thus proved.

We say that two processes $\{x_t'\}$ and $\{x_t''\}$ are *independent*, if the random variables y' and y'' are independent whenever $y' \in [x_t']$ and $y'' \in [x_t'']$.

Now we shall prove the decomposition theorem.

THEOREM 1.1. *Each stationary process admitting a prediction is the sum of two independent stationary processes admitting a prediction, one deterministic and the other completely nondeterministic.*

PROOF. Let $\{x_t\}$ be a stationary process admitting a prediction and let A_t be its predictors. The limit operator $A_{-\infty}$ defined by lemma 1.2. satisfies, in view of lemma 1.1, the equation

$$A_a A_{-\infty} = A_{-\infty} A_a = A_{-\infty}, \qquad (-\infty < a < \infty). \tag{1.11}$$

Consequently,

$$(1 - A_{-\infty})^2 = 1 - A_{-\infty}, \tag{1.12}$$

where 1 is the unit operator. Setting $x_t' = A_{-\infty}x_t$ and $x_t'' = (1 - A_{-\infty})x_t$, we have the formula $x_t = x_t' + x_t''$. Moreover,

$$\begin{gathered}[x_t'] = A_{-\infty}[x_t], \qquad [x_t''] = (1 - A_{-\infty})[x_t], \\ [x_t' \colon t \le 0] = A_{-\infty}[x_t \colon t \le 0], \qquad [x_t'' \colon t \le 0] = (1 - A_{-\infty})[x_t \colon t \le 0].\end{gathered} \tag{1.13}$$

By (iii) and (1.13), the processes $\{x_t'\}$ and $\{x_t''\}$ are independent.

Further, $T_t x_0' = x_t'$ and $T_t x_0'' = x_t''$. Thus both processes $\{x_t'\}$ and $\{x_t''\}$ are stationary. It is very easy to verify that A_0, restricted to $[x_t']$ and $[x_t'']$, is a predictor of $\{x_t'\}$ and $\{x_t''\}$, respectively. By (1.11) and (1.13), $A_0 x = x$ for all $x \in [x_t']$. Consequently, the process $\{x_t'\}$ is deterministic. If $y \in [x_t'']$, then by (1.11), (1.12), and (1.13), $A_t y = (A_t - A_{-\infty})y$, whence the relation $\lim_{t\to-\infty} A_t y = 0$ follows. Thus the process $\{x_t''\}$ is completely nondeterministic, which completes the proof.

The next section will be devoted to the study of stochastic measures which will be used in the representation of completely nondeterministic processes.

2. Stochastic measures

Throughout this section X will denote an arbitrary Banach space consisting of random variables x with zero mean and with the norm $\|x\| = E|x|$.

An X-valued *stochastic measure* M is a function defined on the ring $\mathbf{R}$ of all bounded Borel subsets of the real line, having values in the space X, and such that for disjoint sets $E_1, E_2, \cdots, E_n \in \mathbf{R}$, the random variables $M(E_1), M(E_2), \cdots, M(E_n)$ are independent and $M(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} M(E_j)$ for every sequence $E_1, E_2, \cdots$ of disjoint sets in $\mathbf{R}$ whose union is also in $\mathbf{R}$.

By (1.1), for any stochastic measure M we have the inequality

$$\|M(E_1)\| \le \|M(E_2)\| \qquad \text{if} \quad E_1 \subset E_2. \tag{2.1}$$

Hence, and from general results concerning vector-valued measures ([2], lemma 4, p. 320 and lemma 5, p. 321), it follows that to every stochastic measure M there corresponds a nonnegative Borel measure μ on $\mathbf{R}$ such that

$$\mu(E) \le \|M(E)\|, \qquad (E \in \mathbf{R}) \tag{2.2}$$

and

(2.3) $\|M(E)\| \to 0$ whenever the sets E are bounded in common and $\mu(E) \to 0$.

We now proceed to the definition of integration of scalar functions with respect to the stochastic measure M. An M-null set is a countable union of subsets of sets $E \in \mathbf{R}$ with $M(E) = 0$. Of course, this is the same as a μ-null set. The term M-almost everywhere refers to the complement of an M-null set, and is hence synonymous with the term μ-almost everywhere. A scalar valued function defined on the real line is said to be simple if it is a finite linear combination of indicators of sets in $\mathbf{R}$.

If f is the simple function $\sum_{j=1}^{n} c_j \chi_{E_j}$, where $E_1, E_2, \cdots, E_n \in \mathbf{R}$, then the integral of f over a set $E \in \mathbf{R}$ is defined by the equation

$$\int_E f(u)M(du) = \sum_{j=1}^{n} c_j M(E \cap E_j). \tag{2.4}$$

A scalar valued function f is said to be integrable over the real line with respect to M if there exists a sequence $\{f_n\}$ of simple functions convergent to f, M-almost everywhere such that the sequence $\{\int_{E_n} f_n(u)M(du)\}$ converges in the norm of X for each increasing sequence $E_1 \subset E_2 \subset \cdots$ of sets in $\mathbf{R}$. The limit of this sequence of integrals is defined to be the integral of f over the set $E = \bigcup_{n=1}^{\infty} E_n$ with respect to M, in symbols

$$\int_E f(u)M(du) = \lim_{n\to\infty} \int_{E_n} f_n(u)M(du). \tag{2.5}$$

This definition is a slight modification of a commonly used definition of integration with respect to a vector valued measure (see [2], p. 323) and coincides with it if $E \in \mathbf{R}$. One can prove that the integral in question is an unambiguously defined element of the space X.

By $L(M)$ we shall denote the space of all real-valued functions integrable over the real line with respect to the stochastic measure M. Obviously, $L(M)$ is a linear space. In the sequel we shall identify functions in $L(M)$ equal M-almost everywhere. We shall prove that $L(M)$ is a Banach space under a suitably chosen norm.

LEMMA 2.1. *If $f_1, f_2, \cdots, f_n \in L(M)$ and $E_1, E_2, \cdots, E_n$ are disjoint Borel sets, then the random variables*

$$\int_{E_1} f_1(u)M(du), \qquad \int_{E_2} f_2(u)M(du), \cdots, \int_{E_n} f_n(u)M(du) \tag{2.6}$$

are independent.

PROOF. For simple functions the result is clear. In the general case it follows from the convergence theorem for independent random variables and the definition of the integral.

LEMMA 2.2. *If $E_0 \in \mathbf{R}$, $f \in L(M)$ and $|f(u)| \geq c > 0$ on E_0, then $\|M(E_0)\| \leq 4c^{-1}\left\|\int_{E_0} f(u)M(du)\right\|$.*

PROOF. Put $M_0(E) = \int_E f(u)M(du)$. Let F be a linear functional in X. Denoting by V_0 and V the variation of the scalar measures $F(M_0(E))$ and $F(M(E))$,

respectively, we have the formula $V_0(E) = \int_E |f(u)|V(du)$ for every set $E \in \mathbf{R}$ (see [2], p. 114). Consequently,

$$V_0(E_0) \geq cV(E_0) \geq c|F(M(E_0))|. \tag{2.7}$$

Further, by lemma 5 in ([2], p. 97), we have the inequality

$$V_0(E_0) \leq 4 \sup_{E \subset E_0} |F(M_0(E))| \leq 4\|F\| \sup_{E \subset E_0} \|M_0(E)\|. \tag{2.8}$$

Since by (2.1) and lemma 2.1 $\sup_{E \subset E_0} \|M_0(E)\| = \|M_0(E_0)\|$, the inequalities (2.7) and (2.8) imply $c|F(M(E_0))| \leq 4\|F\| \|M_0(E_0)\|$ for all linear functionals F. Hence, we get the inequality

$$\|M(E_0)\| = \sup_{\|F\|=1} |F(M(E_0))| \leq 4c^{-1}\|M_0(E_0)\|, \tag{2.9}$$

which completes the proof.

By $[M]$ we shall denote the subspace of X spanned by all random variables $M(E)$, $(E \in \mathbf{R})$.

THEOREM 2.1. *For every stochastic measure M the equation $[M] = \{\int_{-\infty}^{\infty} f(u)M(du) : f \in L(M)\}$ holds. Moreover, each element of $[M]$ is uniquely representable as an integral $\int_{-\infty}^{\infty} f(u)M(du)$.*

PROOF. The inclusion $[M] \supset \{\int_{-\infty}^{\infty} f(u)M(du : f \in L(M)\}$ is evident. To prove the converse inclusion it suffices to show that the linear manifold $\{\int_{-\infty}^{\infty} f(u)M(du) : f \in L(M)\}$ is closed in X. Suppose that $f_1, f_2, \cdots \in L(M)$ and the sequence of integrals $\{\int_{-\infty}^{\infty} f_n(u)M(du)\}$ converges in X. By (1.1) and lemma 2.1, for any Borel set E we have the inequality

$$\left\|\int_E (f_n(u) - f_m(u))M(du)\right\| \leq \left\|\int_{-\infty}^{\infty} (f_n(u) - f_m(u))M(du)\right\|. \tag{2.10}$$

Denoting by μ the nonnegative measure satisfying (2.2) and (2.3) and setting $E_{nm}(c) = \{u : |f_n(u) - f_m(u)| \geq c\}$ for any positive constant c, we have, by virtue of lemma 2.2 and (2.10), the inequality

$$\mu(E \cap E_{nm}(c)) \leq \|ME \cap E_{nm}(c))\| \leq 4c^{-1}\left\|\int_{-\infty}^{\infty} (f_n(u) - f_m(u))M(du)\right\| \tag{2.11}$$

for all $E \in \mathbf{R}$ and $n, m = 1, 2, \cdots$. Hence, it follows that the sequence $\{f_n\}$ is fundamental in measure μ on every set $E \in \mathbf{R}$. Thus there is a μ-measurable function f such that $\lim_{n\to\infty} f_n = f$ in measure μ on every set $E \in \mathbf{R}$. Passing, if necessary, to a subsequence, we may assume that $\{f_n\}$ converges to f μ-almost everywhere, and consequently, M-almost everywhere. Further, for every Borel set E we put $N_n(E) = \int_E f_n(u)M(du)$, $(n = 1, 2, \cdots)$. Of course, N_n is a stochastic measure whose domain is the field of all Borel subsets of the real line. Moreover, by (2.10), for each set E the sequence $\{N_n(E)\}$ is fundamental in X, and consequently, converges to an element $N(E)$ of X. By the generalized Nikodým theorem (see [2], p. 321), the set function N is a stochastic measure defined on the field of all Borel subsets of the real line. Moreover, by Vitali-Hahn-Saks theorem ([2], p. 158), $\lim_{n\to\infty} N_n(E_n) = 0$ whenever $\lim_{n\to\infty} \mu(E_n) = 0$ and

$\bigcup_{n=1}^{\infty} E_n \in \mathbf{R}$. Hence, and from the convergence theorem in ([2], p. 325), it follows that the function f is M-integrable over every set $E \in \mathbf{R}$ and

$$N(E) = \lim_{n \to \infty} \int_E f_n(u) M(du) = \int_E f(u) M(du). \tag{2.12}$$

Since by lemma 2.1 the integrals $\int_{E_n} f(u)M(du)$ are independent for disjoint sets $E_1, E_2, \cdots \in \mathbf{R}$ and the series $\sum_{n=1}^{\infty} \int_{E_n} f(u)M(du)$ is convergent in X to $N(\bigcup_{n=1}^{\infty} E_n)$, we infer that the function f is M-integrable over the whole line and that the limit $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(u)M(du) = \int_{-\infty}^{\infty} f(u)M(du)$. Thus the set $\{\int_{-\infty}^{\infty} g(u)M(du) \colon g \in L(M)\}$ is closed in X, which completes the proof of the first statement.

The uniqueness of the integral representation for elements of $[M]$ is a consequence of lemma 2.2. Indeed, if $\int_{-\infty}^{\infty} f(u)M(du) = \int_{-\infty}^{\infty} g(u)M(du)$ and $E(c) = \{u \colon |f(u) - g(u)| \geq c\}$, $(c > 0)$, then lemma 2.2 and a reasoning based on an analogue of inequality (2.10) yield the formula $M(E \cap E(c)) = 0$ for all sets $E \in \mathbf{R}$. Thus, the set $\{u \colon f(u) \neq g(u)\}$ is an M-null set which completes the proof.

As a consequence of theorem 2.1 we get the following corollary.

COROLLARY. *The space $L(M)$ becomes a Banach space under the norm* $\|f\|_0 = \|\int_{-\infty}^{\infty} f(u)M(du)\|$.

Let $\mathbf{R}_0$ be the class of all bounded, semiclosed intervals of the form $(a, b]$. An X-valued function N defined on $\mathbf{R}_0$ is said to be a stochastic interval function if the random variables $N(I_1), N(I_2), \cdots, N(I_n)$ are independent for every system $I_1, I_2, \cdots, I_n$ of disjoint intervals from $\mathbf{R}_0$, $N(J_1 \cup J_2) = N(J_1) + N(J_2)$ whenever J_1 and J_2 are disjoint with $J_1 \cup J_2 \in \mathbf{R}_0$ and $\lim_{c\to b+} N((a, c]) = N((a, b])$ for all intervals $(a, b]$. It is clear that each stochastic measure on $\mathbf{R}$ induces a stochastic interval function on $\mathbf{R}_0$. In the investigation of completely nondeterministic processes the following extension theorem will be used.

THEOREM 2.2. *If N is an X-valued stochastic interval function on $\mathbf{R}_0$, then there is a unique X-valued stochastic measure M on $\mathbf{R}$ such that $M(E) = N(E)$ whenever $E \in \mathbf{R}_0$.*

PROOF. Setting $N(\bigcup_{j=1}^{n} I_j) = \sum_{j=1}^{n} N(I_j)$ for disjoint intervals $I_1, I_2, \cdots, I_n \in R_0$, we extend the function N onto the ring $\mathbf{R}_*$ consisting of all finite unions of intervals from $\mathbf{R}_0$. We shall prove that N is countably additive on $\mathbf{R}_*$.

First we note that, in view of (1.1), the inequality

$$\|N(E_1)\| \leq \|N(E_2)\| \qquad \text{for} \quad E_1 \subset E_2 \tag{2.13}$$

holds. Further, for any linear functional F in X we define a scalar-valued set function $N_F(E) = F(N(E))$ on $\mathbf{R}_*$. Of course, N_F is finitely additive on $\mathbf{R}_*$ and, by (2.13), is bounded on every finite interval. Moreover, $\lim_{c\to b+} N_F((a, c]) = N_F((a, b])$. Hence, it follows that N_F is of bounded variation on every finite interval (see [2], p. 97). Consequently, it is countably additive on $\mathbf{R}_*$. Let $E_1, E_2, \cdots$ be a sequence of disjoint sets from $\mathbf{R}_*$ with $\bigcup_{n=1}^{\infty} E_n \in \mathbf{R}_*$. By (2.13) we have the inequality

$$\left\| \sum_{n=1}^{k} N(E_n) \right\| \leq \left\| N\left(\bigcup_{n=1}^{\infty} E_n \right) \right\|, \qquad (k = 1, 2, \cdots). \tag{2.14}$$

Hence, and from the independence of $N(E_1), N(E_2), \cdots$, it follows that the series $\sum_{n=1}^{\infty} N(E_n)$ converges in X (see [1], p. 338). Since

$$F\left(\sum_{n=1}^{\infty} N(E_n) \right) = \sum_{n=1}^{\infty} F(N(E_n)) = F\left(N\left(\bigcup_{n=1}^{\infty} E_n \right) \right) \tag{2.15}$$

for all linear functionals F in X, we have the equation $\sum_{n=1}^{\infty} N(E_n) = N(\bigcup_{n=1}^{\infty} E_n)$, which shows that N is countably additive on $\mathbf{R}_*$.

Further, if $E_1, E_2, \cdots$ is an arbitrary sequence of disjoint sets from $\mathbf{R}_*$ with the union belonging to $\mathbf{R}$, then, by (2.13), we have the inequality

$$\left\| \sum_{n=1}^{k} N(E_n) \right\| \leq \|N(I)\|, \qquad (k = 1, 2, \cdots) \tag{2.16}$$

where I is an interval from $\mathbf{R}_0$ containing all the sets $E_1, E_2, \cdots$. Hence, and from the independence of $N(E_1), N(E_2), \cdots$, we get the convergence of the series $\sum_{n=1}^{\infty} N(E_n)$ in X (see [1], p. 338). Consequently, by Prékopa's extension theorem ([15], theorem 3.2, p. 243 and section 7) there is a unique random valued set function M on $\mathbf{R}$ such that $M(E) = N(E)$ whenever $E \in \mathbf{R}_*$, and for any sequence $E_1, E_2, \cdots$ of disjoint sets from $\mathbf{R}$ with the union in R, the random variables $M(E_1), M(E_2), \cdots$ are independent and

$$M\left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} M(E_n), \tag{2.17}$$

where the series converges with probability 1. Since $M(E)$ and $M(I \backslash E)$ are independent and $M(E) + M(I \backslash E) = N(I)$ whenever $I \in \mathbf{R}_*$ and $E \subset I$, the random variables $|M(E)|$, $(E \in \mathbf{R})$ have a finite expectation. Thus, by theorem 5.2 in ([1], p. 339), the set function $M(E) - EM(E)$ is a stochastic measure on $\mathbf{R}$ whose restriction to $\mathbf{R}_*$ coincides with $N(E)$. Now, taking into account the uniqueness of the extension M, we infer that $EM(E) = 0$. Hence, by simple reasoning, we get the relation $M(E) \in X$ for all $E \in \mathbf{R}$. Thus M is an X-valued stochastic measure, which completes the proof.

3. Homogeneous stochastic measures

Let $\{T_t\}$ be a one-parameter strongly continuous group of operators in X preserving the probability distribution. An X-valued stochastic measure M is said to be *$\{T_t\}$-homogeneous* if for each set $E \in \mathbf{R}$ the equation $T_t M(E) = M(E + t)$, $(-\infty < t < \infty)$ holds. Here $E + t$ denotes the set $\{u + t : u \in E\}$. It is clear that each X-valued $\{T_t\}$-homogeneous stochastic measure M induces an X-valued homogeneous stochastic process $\{y_t\}$ with independent increments, continuous in the sense of mean convergence and such that

$$M((a, b]) = y_b - y_a, \qquad (a \leq b). \tag{3.1}$$

Conversely, by theorem 2.2, for any X-valued homogeneous stochastic process

$\{y_t\}$ with independent values and continuous in the sense of mean convergence, formula (3.1) uniquely determines an X-valued $\{T_t\}$-homogeneous stochastic measure M.

Let M be an X-valued $\{T_t\}$-homogeneous stochastic measure. By the Lévy-Khintchine representation of infinitely divisible distributions, the logarithm of the characteristic function $h(v, E)$ of $M(E)$, $(E \in \mathbf{R})$ is given by the formula

$$\log h(v, E) = ic|E|v + |E| \int_{-\infty}^{\infty} \left(e^{ivu} - 1 - \frac{ivu}{1 + u^2}\right) \frac{1 + u^2}{u^2} \, dG(u), \tag{3.2}$$

where G is a bounded monotone nondecreasing continuous on the right function with $G(-\infty) = 0$, c is a real constant, and $|E|$ denotes the Lebesgue measure of the set E (see [1], p. 419). Further,

$$\|M(E)\| = E|M(E)| = \frac{2}{\pi} \int_0^{\infty} \frac{1 - Reh(v, E)}{v^2} \, dv \tag{3.3}$$

(see [6], theorem 4.1, p. 274). One can easily prove that $E|M(E)|$ is finite if and only if $\int_{-\infty}^{\infty} |u| \, dG(u)$ is finite. Moreover, $EM(E) = 0$ if and only if $c = -\int_{-\infty}^{\infty} u \, dG(u)$. Thus, by (3.2), $h(v, E)$ is the characteristic function of $M(E)$ for an X-valued $\{T_t\}$-homogeneous stochastic measure M if and only if

$$\log h(v, E) = |E| \int_{-\infty}^{\infty} \left(e^{ivu} - 1 - \frac{ivu}{1 + u^2} - \frac{ivu^3}{1 + u^2}\right) \frac{1 + u^2}{u^2} \, dG(u), \tag{3.4}$$

where G is a monotone nondecreasing continuous on the right function with $G(-\infty) = 0$ and $\int_{-\infty}^{\infty} |u| \, dG(u) < \infty$. Moreover, formula (3.4) determines the function G uniquely.

From (3.3) and (3.4) it follows that $M(E) = 0$ if and only if $|E| = 0$ for any measure M which is not identically equal to 0. Thus, for such measures the class of M-null sets coincides with the class of all sets of Lebesgue measure 0.

In this section we shall give a complete description of the spaces $L(M)$ for $\{T_t\}$-homogeneous stochastic measures M. First of all, we shall quote the definition of Orlicz spaces which are a natural generalization of the L^p-spaces (see [9], [11], and [14]).

Let Φ be a monotone nondecreasing and continuous for $u \geq 0$ function vanishing only at $u = 0$ and tending to ∞ as $u \to \infty$. Two such functions Φ and Ψ are called equivalent, in symbols $\Phi \sim \Psi$, if $a_1\Phi(b_1 u) \leq \Psi(u) \leq a_2\Phi(b_2 u)$ for all $u \geq 0$ and for some positive constants a_1, a_2, b_1, and b_2.

Let $\Lambda(\Phi)$ be the class of all Lebesgue measurable functions f defined on the real line for which the integral $\int_{-\infty}^{\infty} \Phi(|f(u)|) \, du$ is finite. Moreover, we denote by $\Lambda^*(\Phi)$ the class of all functions f such that $af \in \Lambda(\Phi)$, a being a positive number, in general depending on f. The space $\Lambda^*(\Phi)$ is linear and $\Lambda(\Phi)$ is its convex subset. In the space $\Lambda^*(\Phi)$, a nonhomogeneous norm can be defined as follows:

$$\|f\|_{\Phi} = \inf \left\{c : \int_{-\infty}^{\infty} \Phi(c^{-1}|f(u)|) \, du \leq c\right\}. \tag{3.5}$$

The space $\Lambda^*(\Phi)$ becomes a complete linear metric space under this norm and is called *a generalized Orlicz space*. If, in addition, the function Φ is convex, then $\Lambda^*(\Phi)$ is called *an Orlicz space*. In this case, $\Lambda^*(\Phi)$ is a Banach space or, more precisely, a homogeneous norm equivalent to $\|\ \|_\Phi$ can be introduced in $\Lambda^*(\Phi)$. Of course, two equivalent functions Φ and Ψ define the same generalized Orlicz spaces.

We say that the function Φ satisfies condition $(*)$ if there is a positive constant c such that

$$u^2\Phi(v) \leq cv^2\Phi(u) \qquad \text{for all} \quad v \geq u \geq 0 \tag{3.6}$$

and

$$\lim_{u\to 0+} \frac{\Phi(u)}{u} = 0. \tag{3.7}$$

It is clear that the equivalence relation preserves condition $(*)$. It is very easy to verify that the functions u^p $(1 < p \leq 2)$, $(1 + u) \log (1 + u) - u$, and

$$\Phi(u) = \begin{cases} u^2 & \text{if} \quad 0 \leq u \leq 1 \\ 2u - 1 & \text{if} \quad u > 1 \end{cases} \tag{3.8}$$

satisfy condition $(*)$.

In the sequel the stochastic measure identically equal to 0 will be called trivial. The following theorem gives a complete description of the spaces $L(M)$ for nontrivial $\{T_t\}$-homogeneous stochastic measures M.

THEOREM 3.1. *For every nontrivial $\{T_t\}$-homogeneous stochastic measure M there exists a convex function Φ satisfying condition $(*)$ such that $L(M) = \Lambda^*(\Phi)$, and consequently, $L(M)$ is an Orlicz space. Conversely, to every convex function Φ satisfying condition $(*)$ there corresponds a nontrivial $\{T_t\}$-homogeneous stochastic measure M such that $L(M) = \Lambda^*(\Phi)$. Moreover, if G is the Lévy-Khintchine function corresponding to M, then*

$$\Phi(u) \sim u \int_{-\infty}^{\infty} \min (u, |v|^{-1})(1 + v^2)\, dG(v). \tag{3.9}$$

PROOF. First we note that, without loss of generality, we may restrict ourselves to symmetrically distributed stochastic measures. In fact, given an arbitrary $\{T_t\}$-homogeneous stochastic measure M, we put $M_0(E) = M(E) - M'(E)$, $(E \in \mathbf{R})$, where M' and M are independently and identically distributed. It is easy to verify that M_0 is a symmetrically distributed $\{T_t\}$-homogeneous stochastic measure and $L(M_0) = L(M)$.

Let M be a nontrivial symmetrically distributed $\{T_t\}$-homogeneous stochastic measure and G its Lévy-Khintchine function. Put

$$H(v) = \int_{-\infty}^{\infty} (1 - \cos vu) \frac{1 + u^2}{u^2}\, dG(u). \tag{3.10}$$

From (3.4) it follows that the characteristic function $h(v, f)$ of the integral $\int_{-\infty}^{\infty} f(u)M(du)$, $(f \in L(M))$ is given by the formula

$$h(v,f) = \exp\left(-\int_{-\infty}^{\infty} H(v|f(u)|)\,du\right). \tag{3.11}$$

Moreover, by theorem 4.1 in ([6], p. 274),

$$\|f\|_0 = \left\|\int_{-\infty}^{\infty} f(u)M(du)\right\| = \frac{2}{\pi}\int_0^{\infty} \frac{1-h(v,f)}{v^2}\,dv. \tag{3.12}$$

Setting

$$\Psi(u) = u\int_{-\infty}^{\infty}\left(\int_0^{u|v|} \frac{1-\cos w}{w^2}\,dw\right)\frac{1+v^2}{|v|}\,dG(v), \tag{3.13}$$

we obtain a monotone nondecreasing continuous function for $u \geq 0$ vanishing at the origin only and tending to ∞ as $u \to \infty$. One can prove the existence of a positive constant c such that the inequality

$$y^{-1}\int_0^{y} \frac{1-\cos w}{w^2}\,dw \leq cx^{-1}\int_0^{x} \frac{1-\cos w}{w^2}\,dw \tag{3.14}$$

is true for all $y \geq x \geq 0$. Hence, and from (3.13), we get the inequality $v^{-2}\Psi(v) \leq cu^{-2}\Psi(u)$ for all $v \geq u \geq 0$. Consequently, the function Ψ satisfies inequality (3.6). Moreover, by (3.13), $\lim_{u\to 0+} \Psi(u)/u = 0$. Thus the function Ψ satisfies condition (*).

Now we shall prove the equation $L(M) = \Lambda^*(\Psi)$. Given a positive number a and a function $f \in L(M)$, we put $m(a,f) = \min_{0\leq v\leq a} h(v,f)$. Taking into account (3.11) and (3.12), we get the inequality

$$\begin{aligned}\|f\|_0 &\geq \frac{2}{\pi}\int_0^{a} \frac{1-h(v,f)}{v^2}\,dv \geq \frac{2m(a,f)}{\pi}\int_0^{a} v^{-2}\int_{-\infty}^{\infty} H(v|f(u)|)\,du\,dv \\ &= \frac{2m(a,f)}{\pi a}\int_{-\infty}^{\infty} \Psi(a|f(u)|)\,du.\end{aligned} \tag{3.15}$$

Consequently, $f \in \Lambda^*(\Psi)$. Moreover, by (3.5),

$$\|f\|_\Psi \leq a^{-1} \qquad \text{whenever} \quad \|f\|_0 \leq \frac{2m(a,f)}{\pi a^2}. \tag{3.16}$$

Since the mean convergence implies the convergence in probability, we infer that $\lim_{n\to\infty} m(a,f_n) = 1$ for every $a > 0$ whenever $\lim_{n\to\infty} \|f_n\|_0 = 0$. Consequently, by (3.16), the convergence in the norm $\|\ \|_0$ implies the convergence in the norm $\|\ \|_\Psi$ in $L(M)$. Further, from (3.11) and (3.12), by a simple computation, we get the inequality

$$\begin{aligned}\|f\|_0 &\leq \frac{2}{\pi}\int_0^{a} \frac{1-h(v,f)}{v^2}\,dv + \frac{4}{\pi a} \leq \frac{2}{\pi}\int_0^{a} v^{-2}\int_{-\infty}^{\infty} H(v|f(u)|)\,du\,dv \\ &\quad + \frac{4}{\pi a} \leq \frac{2}{\pi a}\int_{-\infty}^{\infty} \Psi(a|f(u)|)\,du + \frac{4}{\pi a},\end{aligned} \tag{3.17}$$

which, by (3.5), implies the relation

$$\|f\|_0 \leq \frac{2}{\pi}\|f\|_\Psi^2 + \frac{4}{\pi}\|f\|_\Psi. \tag{3.18}$$

Thus both norms $\| \ \|_0$ and $\| \ \|_\Psi$ are equivalent in $L(M)$, and consequently, $L(M)$ is complete in the norm $\| \ \|_\Psi$.

From condition $(*)$ we get for all $u \geq 0$ the inequality $\Psi(2u) \leq 4c\Psi(u)$, that is, the Δ_2-condition for Ψ. Consequently, the set of simple functions vanishing outside a finite interval is dense in the space $\Lambda^*(\Psi)$ (see [11], theorems 3.5 and 3.52, p. 155 and theorem 3.53, p. 156). Since all simple functions vanishing outside a finite interval are M-integrable, the set $L(M)$ is dense in $\Lambda^*(\Psi)$. Consequently, $L(M) = \Lambda^*(\Psi)$ because of completeness of $L(M)$ in the norm $\| \ \|_\Psi$. Further, the norm $\| \ \|_0$ equivalent to $\| \ \|_\Psi$ is homogeneous. Thus, by the Mazur-Orlicz theorem ([14], theorem 6, p. 119; see also [13], theorem 2.3, pp. 110–111), the function Ψ is equivalent to a monotone nondecreasing continuous convex function Φ. Of course, the function Φ also satisfies condition $(*)$, and $L(M)$ is the Orlicz space $\Lambda^*(\Phi)$. Moreover, from (3.13), by standard arguments, we obtain (3.9).

Now suppose that Φ is a monotone nondecreasing convex continuous function satisfying condition $(*)$ and tending to ∞ as $u \to \infty$. Then the function $Q_0(u) = \Phi(u)/u$ is monotone nondecreasing and continuous for $u \geq 0$. Moreover, by condition $(*)$, $Q_0(0) = 0$ and $(Q_0(v)/v) \leq c(Q_0(u)/u)$ whenever $v \geq u \geq 0$. Consequently, by theorem 2.7 in ([12], p. 333), there exists a concave nondecreasing continuous function Q with $Q(0) = 0$ equivalent to Q_0. Let q be the right-sided derivative of Q. Since $Q(u) \geq uq(u)$, we have the formula $\lim_{u\to 0+} uq(u) = 0$. Moreover, $q(u)$ tends to a finite limit as $u \to \infty$. Hence, it follows that both integrals

$$\int_0^\infty \frac{u^2}{1+u^2}\, dq(u) \quad \text{and} \quad \int_0^\infty \frac{u}{1+u^2}\, dq(u) \tag{3.19}$$

are finite. Put

$$a = \lim_{u\to\infty} q(u), \qquad b = -\tfrac{1}{2}\int_0^\infty \frac{u^2}{1+u^2}\, dq(u), \tag{3.20}$$

and

$$G(v) = \begin{cases} a + b - \tfrac{1}{2}\displaystyle\int_{v^{-1}}^\infty \frac{u^2}{1+u^2}\, dq(u) & \text{if} \quad v \geq 0, \\ b + \tfrac{1}{2}\displaystyle\int_{|v|^{-1}}^\infty \frac{u^2}{1+u^2}\, dq(u) & \text{if} \quad v < 0. \end{cases} \tag{3.21}$$

The function G is bounded monotone nondecreasing and $G(-\infty) = 0$. Moreover, the finiteness of $\int_0^\infty (u/1 + u^2)\, dq(u)$ implies the existence of the absolute moment $\int_{-\infty}^\infty |u|\, dG(u)$. Consequently, G is a Lévy-Khintchine function for a nontrivial $\{T_t\}$-homogeneous stochastic measure M. By simple computations we get the formula

$$u \int_{-\infty}^\infty \min(u, |v|^{-1})(1 + v^2)\, dG(v) = uq(u) \sim \Phi(u), \tag{3.22}$$

which, by (3.9), gives the equation $L(M) = \Lambda^*(\Phi)$. The theorem is thus proved.

It should be noted that the function Φ defined by (3.8) corresponds to Poisson

stochastic measures. Moreover, it corresponds to stochastic measures for which both integrals $\int_{-\infty}^{\infty} u^2\, dG(u)$ and $\int_{-\infty}^{\infty} dG(u)/|u|$ are finite. Further, the function $\Phi(u) = u^p$, $(1 < p \leq 2)$ corresponds to a stable stochastic measure with exponent p.

4. Completely nondeterministic processes

Throughout this section $\{x_t\}$ will denote a completely nondeterministic process. The process, identically equal to 0, is obviously completely nondeterministic. It will be called trivial.

The predictors $\{A_t\}$ of $\{x_t\}$ define the family $\{A(I)\}$ $(I \in \mathbf{R}_0)$ of operators in $[x_t]$ by means of formula (1.6). Taking into account the definition (1.2), we have the equation

$$T_t A(I) = A(I + t)T_t. \tag{4.1}$$

Moreover, by (1.1) and lemma 1.2, for any $x \in [x_t]$,

$$\|A(I)x\| \leq \|A(J)x\| \qquad \text{whenever} \quad I \subset J. \tag{4.2}$$

Since for $t < 0$, $x_0 = A((t, 0])x_0 + A_t x_0$ and $\lim_{t\to-\infty} A_t x_0 = 0$, we have the relation $x_0 \in [A(I)x: I \in \mathbf{R}_0, x \in [x_t]]$. Consequently,

$$[x_t] = [A(I)x: I \in \mathbf{R}_0, x \in [x_t]]. \tag{4.3}$$

LEMMA 4.1. *Let F be a linear functional in $[X_t]$, g a scalar continuous function with $g(0) \neq 0$, and y an element of $[x_t]$. If for all $I, J \in \mathbf{R}_0$ contained in the interval $(0, a]$ the equation*

$$F\left(A(I) \int_{-a}^{a} g(t)T_t A(J)y\, dt\right) = 0 \tag{4.4}$$

holds, then $F(A((0, a])y) = 0$.

PROOF. We note that the integral in (4.4) is taken in the sense of Bochner (see [5], p. 78). Given $\epsilon > 0$, there exists, in virtue of (1.6) and (4.1), a decomposition of the interval $(0, a]$ into disjoint intervals $I_1, I_2, \cdots, I_n$ belonging to $\mathbf{R}_0$ such that

$$\max_{1 \leq j \leq n} \|A(I_j)y\| < \epsilon. \tag{4.5}$$

Moreover, we may assume that the length of these intervals is so small that $|I_j| + h < a$ $(j = 1, 2, \cdots, n)$, where h is a positive number less than a. Put $J_j = (a_j - h, b_j + h] \cap (0, a]$ if $I_j = (a_j, b_j]$. Since, by (1.7) and (4.1), $A(I_j)T_t A(J_j) = 0$ whenever $|t| > |I_j| + h$, we have the formula

$$A(I_j) \int_{|I_j|+h<|t|\leq a} g(t)T_t A(J_j)y\, dt = 0. \tag{4.6}$$

Consequently,

$$\begin{aligned} A(I_j) \int_{-a}^{a} g(t)T_t A(J_j)y\, dt &= A(I_j) \int_{-h}^{h} g(t)T_t A(J_j)y\, dt \\ &\quad + A(I_j) \int_{h<|t|\leq|I_j|+h} g(t)T_t A(J_j)y\, dt. \end{aligned} \tag{4.7}$$

Further, by (1.7) and (4.1), $A(I_j)T_tA((0, a]\backslash J_j) = 0$ whenever $|t| \leq h$, and consequently,

$$A(I_j) \int_{-h}^{h} g(t)T_tA((0, a])y\, dt = A(I_j) \int_{-h}^{h} g(t)T_tA(J_j)y\, dt, \tag{4.8}$$

which together with (4.7) implies the formula

$$\begin{aligned} A((0, a]) \int_{-h}^{h} g(t)T_tA((0, a])y\, dt &= \sum_{j=1}^{n} A(I_j) \int_{-h}^{h} g(t)T_tA(J_j)y\, dt \\ &= \sum_{j=1}^{n} A(I_j) \int_{-a}^{a} g(t)T_tA(J_j)y\, dt - \sum_{j=1}^{n} A(I_j) \int_{h<|t|\leq|I_j|+h} g(t)T_tA(J_j)y\, dt. \end{aligned} \tag{4.9}$$

Hence and from (4.4) we get the equation

$$\begin{aligned} &F\left(A((0, a]) \int_{-h}^{h} g(t)T_tA((0, a])y\, dt\right) \\ &\qquad = -\sum_{j=1}^{n} F\left(A(I_j) \int_{h<|t|\leq|I_j|+h} g(t)T_tA(J_j)y\, dt\right). \end{aligned} \tag{4.10}$$

But, in view of (4.2) and (4.5),

$$\begin{aligned} &\left\|A(I_j) \int_{h<|t|\leq|I_j|+h} g(t)T_tA(J_j)y\, dt\right\| \\ &\qquad \leq \int_{h<|t|\leq|I_j|+h} g(t)\|A(I_j \cap (J_j + t))y\|\, dt \leq \epsilon q|I_j| \end{aligned} \tag{4.11}$$

where $q = \max_{|t|\leq a} |g(t)|$. Consequently,

$$\left|F\left(A((0, a]) \int_{-h}^{h} g(t)T_tA((0, a])y\, dt\right)\right| \leq \epsilon qa\|F\|. \tag{4.12}$$

Since ϵ can be chosen arbitrarily small, the last inequality implies

$$F\left(A((0, a]) \int_{-h}^{h} g(t)T_tA((0, a])y\, dt\right) = 0 \tag{4.13}$$

for all positive numbers h less than a. Hence, dividing by $2h$ and passing to the limit as $h \to 0$, we obtain, in view of the assumption $g(0) \neq 0$, the equation $F(A((0, a])y) = 0$, which completes the proof.

LEMMA 4.2. *Suppose that the process $\{x_t\}$ is nontrivial. The stochastic interval function M_0, defined on $\mathbf{R}_0$ by the formula*

$$M_0((a, b]) = A((a, b]) \int_a^{\infty} e^{-t}T_tx_0\, dt, \tag{4.14}$$

can be extended to an $[x_t]$-valued stochastic measure on $\mathbf{R}$. The class of M_0-null sets coincides with the class of Lebesgue null sets. Moreover, for any interval $I \in \mathbf{R}_0$, the equation

$$[M_0(J): J \in \mathbf{R}_0, J \subset I] = A(I)[x_t] \tag{4.15}$$

holds, and for $E \in \mathbf{R}, f \in L(M_0)$,

$$A(I) \int_E f(u)M_0(du) = \int_{E\cap I} f(u)M_0(du). \tag{4.16}$$

PROOF. Since $A_0x_0 = x_0$, for all numbers c less than a, we have the equation

$$A((a, b]) \int_c^a e^{-t}T_t x_0 \, dt = 0. \tag{4.17}$$

Consequently,

$$M_0((a, b]) = A((a, b]) \int_c^\infty e^{-t}T_t x_0 \, dt, \qquad (c \leq a). \tag{4.18}$$

Hence, in particular, it follows that $M_0(J_1 \cup J_2) = M_0(J_1) + M_0(J_2)$ for disjoint intervals J_1, J_2 with the union in $\mathbf{R}_0$. Moreover, by lemma 1.2, the random variables $M_0(I_1), M_0(I_2), \cdots, M_0(I_n)$ are independent whenever $I_1, I_2, \cdots, I_n$ are disjoint. Further, from (1.2) and (1.6), the relation $\lim_{c \to b+} M_0((a, c]) = M_0((a, b])$ follows. Thus M_0 is really a stochastic interval function on $\mathbf{R}_0$. By theorem 2.2, it can be extended to an $[x_t]$-valued stochastic measure M_0 on $\mathbf{R}$. Moreover, this extension is unique. From (4.1) and (4.18), we get the equations $A(I)M_0(J) = M_0(I \cap J)$, $T_tM_0(I) = e^tM_0(I + t)$ for all $I, J \in \mathbf{R}_0$. Hence, taking into account the uniqueness of the extension of M_0 onto $\mathbf{R}$, we obtain the equations

$$A(I)M_0(E) = M_0(I \cap E), \qquad T_tM_0(E) = e^tM_0(E + t), \qquad (I \in \mathbf{R}_0,\ E \in \mathbf{R}). \tag{4.19}$$

As a consequence of the first equation, we get formula (4.16) for simple functions. The general case can be obtained by an approximation of M_0-integrable functions by simple ones.

Now we shall prove the relation

$$A((a, b]) \int_a^\infty e^{-t}T_t x \, dt \in [M_0] \text{ for all } \quad x \in [x_t : t \leq 0] \quad \text{and} \quad a \leq b. \tag{4.20}$$

First we observe that the set of elements x satisfying (4.20) is a subspace of the space $[x_t]$. If $h \leq 0$, then by (4.18),

$$A((a, b]) \int_a^\infty e^{-t}T_tT_h x_0 \, dt = e^hA((a, b]) \int_{a+h}^\infty e^{-t}T_t x_0 \, dt = e^hM_0((a, b]), \tag{4.21}$$

and consequently, all elements T_hx_0 with $h \leq 0$ satisfy (4.20). Thus all elements $x \in [x_t : t \leq 0]$ satisfy (4.20).

The inclusion $[M_0(J) : J \in \mathbf{R}_0,\ J \subset I] \subset A(I)[x_t]$ is obvious. By the second equation (4.19), it suffices to prove the converse inclusion for intervals I of the form $(0, a]$. Suppose that there is an element y in $A((0, a])\,[x_t]$ which does not belong to $[M_0(J) : J \in \mathbf{R}_0,\ J \subset (0, a]]$. There exists then a linear functional F on $[x_t]$ vanishing on $[M_0(J) : J \in \mathbf{R}_0, J \subset (0, a]]$, and such that $F(y) = 1$. Given two intervals $I, J \in \mathbf{R}_0$ contained in $(0, a]$, we put

$$z(I, J) = e^aA(I) \int_0^\infty e^{-t}T_tT_{-a}A(J)y \, dt. \tag{4.22}$$

Since $T_{-a}A(J)y \in [x_t : t \leq 0]$, we have, by (4.19) and (4.20), the relation

$$z(I, J) \in [M_0(U) : U \in \mathbf{R}_0,\ U \subset (0, a]]. \tag{4.23}$$

Thus

$$F(z(I, J)) = 0. \tag{4.24}$$

Further, from the equation $A((0, a])T_tA(J) = 0$ for $t \geq a$ we get the formula

$$z(I, J) = A(I) \int_{-a}^{a} e^{-t}T_tA(J)y\,dt. \tag{4.25}$$

Hence, by (4.24) and lemma 4.1, we get the equation $F(A((0, a])y) = 0$. But $A((0, a])y = y$, and consequently, $F(y) = 0$, which contradicts the equation $F(y) = 1$. Formula (4.15) is thus proved.

By (4.3) and (4.15), the stochastic measure M_0 is not identically equal to 0. Consequently, the nonnegative measure μ_0 associated with M_0 and satisfying conditions (2.2) and (2.3) does not vanish identically. From (4.19) it follows that the class of M_0-null sets, and consequently, μ_0-null sets is invariant under translations. Thus, it coincides with the class of Lebesgue null sets (see [3], chapter IV, section 5) which completes the proof of the lemma.

THEOREM 4.1. *For each completely nondeterministic process* $\{x_t\}$ *there exists an* $[x_t]$*-valued* $\{T_t\}$*-homogeneous stochastic measure* M *such that for any interval* $I \in \mathbf{R}_0$,

$$[M(J): J \in \mathbf{R}_0, J \subset I] = A(I)[x_t]. \tag{4.26}$$

PROOF. For a trivial process $\{x_t\}$ the trivial measure M satisfies the assertion of the theorem. Suppose that the process $\{x_t\}$ is nontrivial. First we shall prove that there exists an element y_0 belonging to $A((0, 1])\,[x_t]$ such that

$$A((0, 1]) \int_{-1}^{1} T_ty_0\,dt \neq 0. \tag{4.27}$$

Contrary to this, let us suppose that $A((0, 1]) \int_{-1}^{1} T_tz\,dt = 0$ for all $z \in A((0, 1])\,[x_t]$. Given an element $y \neq 0$ in $A((0, 1])\,[x_t]$, there is a linear functional F on $[x_t]$ with $F(y) = 1$. Since for any interval $J \in \mathbf{R}_0$ and contained in $(0, 1]$, $A((0, 1]) \int_{-1}^{1} T_tA(J)y\,dt = 0$, we have, by (1.7), $A(I) \int_{-1}^{1} T_tA(J)y\,dt = 0$ for all intervals $I \in \mathbf{R}_0$ and contained in $(0, 1]$. Hence, by lemma 4.1, $F(A((0, 1])y) = 0$, which, in view of the formula $A((0, 1])y = y$, contradicts the equation $F(y) = 1$. Thus (4.27) holds for an element y_0 in $A((0, 1])\,[x_t]$.

Put

$$M((a, b]) = A((a, b]) \int_{a-1}^{b} T_ty_0\,dt. \tag{4.28}$$

Taking into account the formula $A((0, 1])y_0 = y_0$, we get the equation

$$M((a, b]) = A((a, b]) \int_{c}^{d} T_ty_0\,dt \tag{4.29}$$

for $d \geq b$ and $c \leq a - 1$. Hence, it follows that $M(J_1 \cup J_2) = M(J_1) + M(J_2)$ for disjoint intervals J_1, J_2 with $J_1 \cup J_2 \in \mathbf{R}_0$. Moreover,

$$A(I)M(J) = M(I \cap J) \tag{4.30}$$

for all $I, J \in \mathbf{R}_0$. The relation $\lim_{c\to b+} M((a, c]) = M((a, b])$ is evident. Conse-

quently, by lemma 1.2, M is a stochastic interval function on $\mathbf{R}_0$. By theorem 2.2, it can be extended to a stochastic measure M on $\mathbf{R}$, and this extension is unique. By (4.1) and (4.28), $T_tM(I) = M(I + t)$ for $I \in \mathbf{R}_0$. Taking into account the uniqueness of the extension we have the formula $T_tM(E) = M(E + t)$ for all $E \in \mathbf{R}$. Thus, the stochastic measure M is $\{T_t\}$-homogeneous. Further, from (4.30), we get the inclusion

$$(4.31) \qquad [M(J): J \in \mathbf{R}_0, J \subset I] \subset A(I)[x_t].$$

Let M_0 be the stochastic measure defined by formula (4.14). From formulas (4.15), (4.16), (4.30), (4.31), and theorem 2.1, we get the existence of a function $g \in L(M_0)$ satisfying the equation

$$(4.32) \qquad M(E) = \int_E g(u)M_0(du)$$

for all $E \in \mathbf{R}$. Since by (4.27) $M((0, 1]) \neq 0$, the class of M-null sets is the class of sets of Lebesgue measure zero, and consequently, coincides with the class of M_0-null sets. Thus, the function g differs from 0 almost everywhere with respect to both measures M and M_0. From (4.32) we get the formula

$$(4.33) \qquad \int_E f(u)M(du) = \int_E f(u)g(u)M_0(du)$$

for all sets $E \in \mathbf{R}$ and all simple functions f. Let $\{f_n\}$ be a sequence of simple functions such that $|f_n(u)| \leq |g(u)|^{-1}$ and $\lim_{n\to\infty} f_n(u) = g(u)^{-1}$ M_0-almost everywhere. Then, by the theorem on dominated convergence (see [2], p. 328) and formula (4.33), we have the relation

$$(4.34) \qquad M_0(I) = \lim_{n\to\infty} \int_I f_n(u)g(u)M_0(du) \in [M(J): J \in \mathbf{R}_0, J \subset I],$$

which, together with (4.15) and (4.31), implies (4.26). The theorem is thus proved.

Now we shall prove a representation theorem for nontrivial completely nondeterministic processes.

THEOREM 4.2. *Let $\{x_t\}$ be a nontrivial completely nondeterministic process. Then there exist an $[x_t]$-valued nontrivial $\{T_t\}$-homogeneous stochastic measure M and a function f belonging to the Orlicz space $L(M)$ such that*

$$(4.35) \qquad [M(J): J \in \mathbf{R}_0, J \subset (-\infty, 0]] = [x_t: t \leq 0]$$

and

$$(4.36) \qquad x_t = \int_{-\infty}^{t} f(u - t)M(du).$$

Conversely, if M is a nontrivial $\{T_t\}$-homogeneous stochastic measure and $f \in L(M)$, then the process (4.36) *is nontrivial and completely nondeterministic, provided equation* (4.35) *holds.*

PROOF. Given a nontrivial completely nondeterministic process $\{x_t\}$, there exists, by theorem 4.1, a nontrivial $[x_t]$-valued $\{T_t\}$-homogeneous stochastic measure M satisfying condition (4.35). Hence, by theorem 2.1, we get the

existence of a function $f \in L(M)$ such that $x_0 = \int_{-\infty}^{0} f(u)M(du)$. Consequently, by the translation property of $\{T_t\}$-homogeneous measures,

$$x_t = T_t x_0 = \int_{-\infty}^{t} f(u-t)M(du). \tag{4.37}$$

Now suppose that M is a nontrivial $\{T_t\}$-homogeneous stochastic measure, that $f \in L(M)$, and that the process $\{x_t\}$ defined by (4.36) satisfies condition (4.35). Then, of course, $[x_t] = [M]$ and, by theorem 2.1, each element $x \in [x_t]$ has an integral representation $x = \int_{-\infty}^{\infty} g(u)M(du)$ where $g \in L(M)$. Put

$$A_0 x = \int_{-\infty}^{0} g(u)M(du). \tag{4.38}$$

By (4.35) the linear operator A_0 transforms $[x_t]$ onto $[x_t: t \leq 0]$. Further, the conditions (i) and (iii) for predictors are obvious. In order to prove (ii), suppose that $x \in [x_t]$ and for every $y \in [x_t: t \leq 0]$ the random variables x and y are independent. Hence, in particular, it follows that the random variables x and $A_0 x$ are independent. Since $x - A_0 x$ and $A_0 x$ are also independent, we infer by simple reasoning that $A_0 x$ is a constant random variable; hence, $A_0 x = 0$. Thus, condition (ii) is also fulfilled, and consequently, A_0 is the predictor for $\{x_t\}$ based on the past up to time $t = 0$. Finally, $A_t x = \int_{-\infty}^{t} g(u)M(du)$, which implies $\lim_{t \to -\infty} A_t x = 0$. Thus the process $\{x_t\}$ is completely nondeterministic. Obviously, it is also nontrivial, which completes the proof of the theorem.

5. Some concluding remarks

A $\{T_t\}$-homogeneous stochastic measure M will be called *a Wiener stochastic measure* if all the random variables $M(E)(E \in R)$ are Gaussian. Of course, a Wiener stochastic measure is induced by a Wiener process. Moreover, a stochastic measure corresponding to a Gaussian completely nondeterministic process is a Wiener stochastic measure. From the Skitovich results ([16], theorem 1, p. 362), it follows that if M is a nontrivial $\{T_t\}$-homogeneous stochastic measure and $\int_a^b f(u)M(du)$, $\int_a^b g(u)M(du)$ are independent, where f and g are continuous functions in a closed interval $[a, b]$ such that fg does not vanish identically, and at least one of the integrals

$$\int_a^b \frac{f^2(u)}{g^2(u)}\, du \quad \text{or} \quad \int_a^b \frac{g^2(u)}{f^2(u)}\, du \tag{5.1}$$

exists, then M is a Wiener stochastic measure. R. G. Laha and E. Lukacs showed ([7], p. 312) that for stochastic measures with a finite variance the assumption concerning the existence of integrals (5.1) can be removed. In the sequel we shall use the following generalization of mentioned results.

THEOREM 5.1. *Let M be a nontrivial $\{T_t\}$-homogeneous stochastic measure and $f, g \in L(M)$. If fg does not vanish M-almost everywhere and $\int_{-\infty}^{\infty} f(u)M(du)$, $\int_{-\infty}^{\infty} g(u)M(du)$ are independent, then M is a Wiener stochastic measure.*

PROOF. By Cramér's decomposition theorem (see [8], p. 271), it suffices to prove the theorem for symmetrically distributed stochastic measures M. Let G be the Lévy-Khintchine function corresponding to M. Put

$$H(v) = \int_{-\infty}^{\infty} (1 - \cos vu) \frac{1 + u^2}{u^2} dG(u). \tag{5.2}$$

Then the characteristic function of the integral $\int_{-\infty}^{\infty} f(u)M(du)$ is given by formula (3.11). Consequently, the independence of the integrals $\int_{-\infty}^{\infty} f(u)M(du)$ and $\int_{-\infty}^{\infty} g(u)M(du)$ implies the equation

$$\int_{-\infty}^{\infty} H(uf(t) + vg(t))\, dt = \int_{-\infty}^{\infty} H(uf(t))\, dt + \int_{-\infty}^{\infty} H(vg(t))\, dt \tag{5.3}$$

for all u and v. Setting

$$\begin{aligned} Q(u, v) &= 2H(u) + 2H(v) - H(u + v) - H(u - v) \\ &= 2 \int_{-\infty}^{\infty} (1 - \cos ux)(1 - \cos vx) \frac{1 + x^2}{x^2} dG(x), \end{aligned} \tag{5.4}$$

we get the inequality

$$Q(u, v) \geq 0 \tag{5.5}$$

for all u and v. Further, from (5.3) we obtain the equation

$$\int_{-\infty}^{\infty} Q(uf(t), vg(t))\, dt = 0 \tag{5.6}$$

for all u and v. Consequently, by (5.5), for every u and v the equation $Q(uf(t), vg(t)) = 0$ holds almost everywhere. Since $fg \neq 0$ on a set of positive Lebesgue measure, and H is a continuous function, we infer that $Q(u, v) = 0$ for all u and v. Thus, by (5.4), the function H satisfies the equation

$$H(u + v) + H(u - v) = 2H(u) + 2H(v). \tag{5.7}$$

The same arguments as in the case of Cauchy functional equation show that the functions $H(u) = cu^2$, where c is a constant, are the only continuous solutions of this equation. Hence, and from (3.11), it follows that the random variables $M(E)$, $(E \in R)$ are Gaussian which completes the proof.

THEOREM 5.2. *Let M_1 and M_2 be $\{T_t\}$-homogeneous stochastic measures and*

$$[M_1] \subset [M_2]. \tag{5.8}$$

If M_2 is not a Wiener stochastic measure, then there are real constants c and a such that $M_1(E) = cM_2(E + a)$ for all sets $E \in \mathbf{R}$.

PROOF. If the stochastic measure M_2 is trivial, then the assertion is obvious. Suppose that M_2 is nontrivial. By (5.8) and theorem 2.1, for any set $E \in \mathbf{R}$ there exists exactly one function $f_E \in L(M_2)$ such that

$$M_1(E) = \int_{-\infty}^{\infty} f_E(u) M_2(du). \tag{5.9}$$

Since the random variables $\int_{-\infty}^{\infty} f_{E_1}(u)M_2(du)$, $\int_{-\infty}^{\infty} f_{E_2}(u)M_2(du)$ are independent

for disjoint sets E_1, E_2 and M_2 is not a Wiener stochastic measure, we have, by theorem 5.1, the equation

$$f_{E_1}f_{E_2} = 0 \qquad \text{whenever} \quad E_1 \cap E_2 = \varnothing. \tag{5.10}$$

Moreover, by the uniqueness of the integral representation,

$$f_{E_1} + f_{E_2} = f_{E_1 \cup E_2} \qquad \text{if} \quad E_1 \cap E_2 = \varnothing. \tag{5.11}$$

Put $S(E) = \{u : f_E(u) \neq 0\}$. The set $S(E)$ is defined up to an M_2-null set, and consequently, up to a Lebesgue null set. In what follows two sets are treated as identical if their symmetric difference is a Lebesgue null set. From (5.10) and (5.11) we get the equations

$$S(E_1) \cap S(E_2) = \varnothing, \qquad S(E_1) \cup S(E_2) = S(E_1 \cup E_2) \tag{5.12}$$

for disjoint sets E_1 and E_2. Further, by (5.10) and (5.11), $f_E = f_F$ on $S(E)$ whenever $E \subset F$. Thus formula (5.9) can be written in the form

$$M_1(E) = \int_{S(E)} f(u) M_2(du), \tag{5.13}$$

where the function f does not depend on E and is M_2-integrable over every set $S(E)$, $(E \in \mathbf{R})$. Moreover, for any number t we have the equation

$$\begin{aligned} \int_{S(E)} f(u) M_2(du) &= M_1(E) = T_t M_1(E - t) \\ &= \int_{S(E-t)} f(u) T_t M_2(du) = \int_{S(E-t)+t} f(u - t) M_2(du). \end{aligned} \tag{5.14}$$

Hence, by the uniqueness of the integral representation, we obtain the formula

$$S(E) = S(E - t) + t. \tag{5.15}$$

Moreover, $f(u) = f(u - t)$ M_2-almost everywhere on $S(E)$, $(E \in \mathbf{R})$. Consequently, the function f is constant M_2-almost everywhere and, by (5.13),

$$M_1(E) = cM_2(S(E)), \qquad (E \in \mathbf{R}), \tag{5.16}$$

where c is a constant. If $c = 0$, then the assertion is obvious. Suppose that M_1 is nontrivial. Then, by formula (3.4), we have the equation $|E| = c_0|S(E)|$, $(E \in \mathbf{R})$, where c_0 is a positive constant. For any simple function $g = \sum_{j=1}^{n} a_j \chi_{E_j}$, $(E_j \in \mathbf{R})$, we put

$$Ug = \sum_{j=1}^{n} a_j \chi_{S(E_j)}. \tag{5.17}$$

By (5.12) and (5.15), the transformation U is linear in the space of simple functions, transforms indicators into indicators, commutes with the translations, and satisfies the equation

$$(Ug_1)(Ug_2) = 0 \qquad \text{if} \quad g_1 g_2 = 0. \tag{5.18}$$

Since $\int_{-\infty}^{\infty} |(Ug)(u)|\, du = c_0^{-1} \int_{-\infty}^{\infty} |g(u)|\, du$, the transformation U can be extended to a linear operator in the L-space over the real line commuting with the translations. Consequently, there is a finite measure μ on the field of Borel subsets of the real line such that

$$(5.19) \qquad (Ug)(v) = \int_{-\infty}^{\infty} g(v-u)\mu(du)$$

(see [5], theorem 21.2.3, p. 568). Since U transforms indicators into indicators, the measure μ is nonnegative. Setting $I_n = [-n, n]$, we have, by (5.18) and (5.19),

$$(5.20) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_{I_1}(x-u)\chi_{I_n\setminus I_1}(x-v)\mu(du)\mu(dv) = 0$$

almost everywhere. Consequently, the equation

$$(5.21) \qquad \chi_{I_1}(x-u)\chi_{I_n\setminus I_1}(x-v) = 0$$

holds for almost every x and $\mu \times \mu$-almost every pair u, v. But the left-hand side of (5.21) is equal to 1 for all $n \geq 1 + |u - v|$ and all numbers x satisfying the inequality $1 + v < x \leq 1 + u$ if $u > v$ and the inequality $u - 1 \leq x < v - 1$ if $v > u$. Thus the product measure $\mu \times \mu$ is concentrated at the diagonal $u = v$, and consequently, the measure μ is concentrated at a single point. Hence, and from (5.19), we get the formula

$$(5.22) \qquad \chi_{S(E)} = U\chi_E = \chi_{E+a}, \qquad (E \in \mathbf{R}),$$

where a is a constant. Thus $S(E) = E + a$, which together with (5.16) implies the assertion of the theorem.

A stationary process $\{x_t\}$ is said to be *indecomposable* if for every decomposition $x_0 = y_0 + z_0$, $(y_0, z_0 \in [x_t])$, for which the processes $\{T_ty_0\}$ and $\{T_tz_0\}$ are independent and completely nondeterministic, at least one component y_0 or z_0 vanishes.

THEOREM 5.3. *Stationary processes admitting a prediction are indecomposable.*

PROOF. For Gaussian processes the result is well-known (see [4], theorem 3, p. 177). Suppose that the process $\{x_t\}$ is not Gaussian and set $y_t = T_ty_0$, $z_t = T_tz_0$. Since the subspace $[x_t: t \leq a]$ is contained in the direct sum of subspaces $[y_t: t \leq a]$ and $[z_t: t \leq a]$ and the processes $\{y_t\}$, $\{z_t\}$ are independent and completely nondeterministic, we have the relation $\bigcap_a [x_t: t \leq a] = \{0\}$. Consequently, the process $\{x_t\}$ is also completely nondeterministic. Let M, M_1, and M_2 be $\{T_t\}$-homogeneous stochastic measures induced by $\{x_t\}$, $\{y_t\}$, and $\{z_t\}$, respectively. Of course, M is not a Wiener stochastic measure and $[M] \supset [M_1]$ and $[M] \supset [M_2]$. Thus, by theorem 5.2, either $[M_j] = [M]$ or $[M_j] = \{0\}$, $(j = 1, 2)$. By the independence of $\{y_t\}$ and $\{z_t\}$ we have the equation $[M_1] \cap [M_2] = \{0\}$. Consequently, either $[M_1] = \{0\}$ or $[M_2] = \{0\}$, which implies that at least one component, y_0 or z_0, vanishes. The theorem is thus proved.

The following theorem gives a characterization of Gaussian completely nondeterministic processes.

THEOREM 5.4. *A nontrivial completely nondeterministic process $\{x_t\}$ is Gaussian if and only if for every $y \in [x_t]$ the process $\{T_ty\}$ admits a prediction.*

PROOF. The necessity of the condition is obvious. In order to prove the sufficiency of this condition, consider the integral representation of $\{x_t\}$ in terms of a nontrivial $\{T_t\}$-homogeneous stochastic measure M. By theorem 3.1, $L(M)$

is an Orlicz space $\Lambda^*(\Phi)$ defined by a convex function Φ satisfying condition (*). Put $g(u) = e^{-u}$ if $u \geq 0$, and $g(u) = 0$ if $u < 0$. Since, by (3.7),

$$\int_{-\infty}^{\infty} \Phi(g(u))\, du = \int_0^1 \frac{\Phi(u)}{u}\, du < \infty, \tag{5.23}$$

the function g belongs to $L(M)$ and, consequently, the element

$$y_0 = \int_{-\infty}^{\infty} g(u)M(du) = \int_0^{\infty} e^{-u}M(du) \tag{5.24}$$

belongs to $[x_t]$. Ptu $y_t = T_t y_0$. By the assumption, the process $\{y_t\}$ admits a prediction. Further, for any $a \leq b$, we have the formula

$$e^{-a}y_a - e^{-b}y_b = \int_a^b e^{-u}M(du). \tag{5.25}$$

First we shall prove that the process y_t is nondeterministic. Suppose the contrary. Then $y_1 \in [y_t\colon t \leq 0]$, and consequently, there is a sequence $z_n = \sum_{j=1}^{k_n} c_{jn} y_{t_j}$, $(t_j \leq 0)$ convergent to y_1. Thus, by (5.25),

$$\begin{aligned}\int_1^2 e^{-u}M(du) &= e^{-1}y_1 - e^{-2}T_1 y_1 = \lim_{n\to\infty} (e^{-1}z_n - e^{-2}T_1 z_n) \\ &= \lim_{n\to\infty} \sum_{j=1}^{k_n} c_{jn} \exp(t_j - 1) \int_{t_j}^{t_j+1} e^{-u}M(du),\end{aligned} \tag{5.26}$$

which shows that

$$\int_1^2 e^{-u}M(du) \in [M(J)\colon J \in \mathbf{R}_0,\ J \subset (-\infty, 1]]. \tag{5.27}$$

On the other hand, the random variable $\int_1^2 e^{-u}M(du)$ and each random variable from $[M(J)\colon J \in \mathbf{R}_0,\ J \subset (-\infty, 1]]$ are independent. Thus, $\int_1^2 e^{-u}M(du) = 0$. Since the stochastic measure M is nontrivial, the last equation contradicts the uniqueness of the integral representation. Thus, the process $\{y_t\}$ is nondeterministic.

By the decomposition theorem 1.1, $y_t = y_t' + y_t''$ where the processes $\{y_t'\}$ and $\{y_t''\}$ are independent, $\{y_t'\}$ is deterministic, and $\{y_t''\}$ is completely nondeterministic. Moreover, $y_t' = T_t y_0'$, $y_t'' = T_t y_0''$, and the component $\{y_t''\}$ does not vanish.

First, let us assume that the component $\{y_t'\}$ also does not vanish. Since $y_0', y_0'' \in [x_t]$, we have the integral representation

$$y_0' = \int_{-\infty}^{\infty} g'(u)M(du), \qquad y_0'' = \int_{-\infty}^{\infty} g''(u)M(du) \tag{5.28}$$

where g', g'' belong to $L(M)$ and do not vanish almost everywhere. Moreover,

$$y_t' = \int_{-\infty}^{\infty} g'(u - t)M(du) \quad \text{and} \quad y_t'' = \int_{-\infty}^{\infty} g''(u - t)M(du). \tag{5.29}$$

It is clear that there are two numbers t_1 and t_2 such that the product $g'(u - t_1)g''(u - t_2)$ does not vanish almost everywhere. Since y_{t_1}' and y_{t_2}'' are

independent, we infer, in view of theorem 5.1, that M is a Wiener stochastic measure, and consequently, $\{x_t\}$ is a Gaussian process.

Finally, suppose that the deterministic component vanishes; then $\{y_t\}$ is a completely nondeterministic process. By the representation theorem 4.2, there is then a nontrivial $\{T_t\}$-homogeneous stochastic measure M' such that $[M'] \subset [M]$ and

$$y_t = \int_{-\infty}^{t} f(u - t)M'(du), \tag{5.30}$$

where $f \in L(M')$. Suppose that M is not a Wiener stochastic measure. Then, by theorem 5.2, $M'(E) = cM(E + a)$ where $c \neq 0$ because the process $\{y_t\}$ is nontrivial. Consequently, by (5.29),

$$y_0 = c \int_{-\infty}^{a} f(u - a)M(du), \tag{5.31}$$

which, by the uniqueness of the integral representation, contradicts (5.24). Thus, M is a Wiener stochastic measure, and consequently, $\{x_t\}$ is a Gaussian process, which completes the proof.

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